Communications on Stochastic Analysis
Vol. 3, No. 1 (2009) 101-117

Serials Publications
www.serialspublications.com

OPTIMAL CONSUMPTION AND PORTFOLIO FOR AN INSIDER IN A MARKET WITH JUMPS

DELPHINE DAVID AND YELIZ YOLCU OKUR

Abstract. We examine a stochastic optimal control problem in a financial market driven by a Lévy process with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. We assume that in the market there are two kinds of investors with different levels of information: an *uninformed agent* whose information coincides with the natural filtration of the price processes and an *insider* who has more information than the uninformed agent. When optimal consumption and investment exist, we identify some necessary conditions and find the optimal strategy by using forward integral techniques. We conclude by giving some examples.

1. Introduction

The consumption-portfolio problem in continuous time market models was first introduced by Merton [22], [23]. He worked with the assumption that stock prices were governed by Markovian dynamics with constant coefficients. This approach is based on stochastic dynamic programming. For a market which consists of only two assets, he formulated the problem of choosing optimal portfolio selection and consumption rules as follows:

$$\max_{(c,X)} E\left[\int_0^T U(c(t),t)dt + g(X(T),T)\right],$$

subject to the budget constraint, $c(t) \geq 0$, $X(t) > 0$ for all $t \in [0,T]$ with $X(0) = x$. Here $X(\cdot)$ represents the wealth process, $c(\cdot)$ is the consumption per unit time, U is assumed to be a strictly concave *utility* function and g is the *bequest valuation* function which is concave in terminal wealth $X(T)$. Recently, many authors have used a martingale representation technique instead of dynamic programming methods: see Cox and Huang [4], [5], Karatzas, Lehoczky and Shreve [15] and Pliska [26]. In incomplete markets, the theory was studied by He and Pearson [12], Karatzas *et al.* [16], Karatzas and Zitkovic [18], Kramkov and Schachermayer [20].

In financial markets a trader is assumed to make her decisions with respect to the information revealed by the market events. This information is assumed to be accessible to everyone. However, in reality some agents have superior information

2000 *Mathematics Subject Classification.* Primary 93E20 91B28; Secondary 60H05 60G57.

Key words and phrases. Insider trading, forward integral, Levy process, portfolio optimization.

* This research was partially supported by grants from AMaMeF (Projects No #2062 and #1850).

about the market. The informed agent possesses information regarding some future movements in stock prices and she has opportunities to gain profits by trading before prices reach equilibrium. Because of this fact, it is more realistic to model stochastic control problems where the information is non-identical. This paper focuses on the characterization of the optimal consumption and portfolio choices of an insider when the market is driven by a Lévy process. We also compare the performance functions of informed and uninformed agents in certain specific cases. These results can be helpful in detecting informed agents.

Karatzas and Pikovsky [17]'s study on stochastic control problems with the presence of insiders is one of the earliest studies regarding to this approach. In this study, they assume that the informed agents maximize their expected logarithmic utility from terminal wealth and consumption in Brownian motion framework. See also Amendinger *et al.* [1], Grorud *et al.* [11] and Elliott *et al.* [8]. In a subsequent, Grorud [10] analyzed the optimal portfolio and consumption choices when prices are driven by a Brownian motion and a compound Poisson process. The first general study of insider trading based on forward integrals (without assuming the enlargement of filtration) was done in Biagini and Øksendal [3]. Thereafter, many authors used Malliavin calculus and forward integration to study the optimal portfolio choices of insiders. See e.g. Biagini and Øksendal [2], Imkeller [13], Kohatsu-Higa and Sulem [19] and León *et al.* [21] for further discussions of the Brownian motion case. An extension of forward integration in to the case of compensated Poisson random measures was proposed by Di Nunno *et al.* [6]. That setting is used in solving the optimal portfolio problem as elaborated by Di Nunno *et al.* [7] and in solving the optimal consumption rate problem as elaborated by Øksendal [25].

In this paper, we extend the results of Di Nunno *et al.* [7] and of Øksendal [25] by considering both the optimal portfolio and consumption rate choices of an insider when her portfolio is allowed to anticipate the future. We formulate the associated optimal control problem as follows:

$$\max_{(c,\pi)} \mathbb{E}\left[\int_0^T e^{-\delta(t)} \ln c(t) dt + K e^{-\delta(T)} \ln X^{(c,\pi)}(T)\right].$$

$\delta(t) \geq 0$ is a given bounded deterministic function representing the discount rate, K is a nonnegative constant representing the weight of the expected utility of the terminal wealth in the performance function, $T > 0$ is a fixed terminal time and $X^{(c,\pi)}(\cdot)$ is the wealth process with control parameters (c, π). As we consider the optimal consumption and the optimal portfolio problems together, the solution for the optimal portfolio is more complex than the studies which consider only the optimization with respect to terminal wealth. Moreover, unlike many studies we do not use the initial enlargement of filtration technique. As a result, we prove that if there exist optimal portfolio and consumption, then the $\mathbb{F}$-Brownian motion $B(\cdot)$ is a $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$-semimartingale where $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \in [0, T]$ and we show that it also holds for Lévy processes (see Theorem 4.5).

This paper is organized as follows. In Section 2, we recall some mathematical preliminaries about forward integrals which are relevant to our calculations. In

Section 3, we introduce the main problem. In Section 4, we characterize the optimal consumption and portfolio choice for the problem introduced in the previous section. In Section 5, we compare the optimal wealth process and the performance function of informed and uninformed agents in certain specific examples using the results in Section 4.

2. Preliminary Notes

In this section, we recall the forward integrals with respect to the Brownian motion and to the compensated Poisson random measure. For further information on the forward integration with respect to the Brownian motion, we refer to Russo and Vallois [27], [28] and [29], Nualart [24], Biagini and Øksendal [2]. For the forward integration with respect to the compensated Poisson random measure we refer to Di Nunno *et al.* [6] and [7].

Let $(\Omega, \mathcal{F}, P)$ be a product of probability space such that

$$(\Omega, P) = (\Omega_B \times \Omega_\eta, P_B \otimes P_\eta)$$

on which are respectively defined a standard Brownian motion $\{B(t)\}_{0\leq t\leq T}$ and a pure jump Lévy process, $\{\eta(t)\}_{0\leq t\leq T}$, such that

$$\eta(t) = \int_0^t \int_{\mathbb{R}} z\tilde{N}(dt, dz),$$

where $\tilde{N}(dt, dz) = (N - \nu_{\mathcal{F}})(dt, dz) = N(dt, dz) - \nu_{\mathcal{F}}(dz)dt$ is a compensated Poisson random measure with Lévy measure $\nu_{\mathcal{F}}$.

Let $\mathcal{B}$ denote the σ-field of Borel sets.

$$\mathcal{F}_t^B := \sigma\{B(s), s \leq t,\, t \in [0, T]\} \vee \mathcal{N}$$

and

$$\mathcal{F}_t^{\tilde{N}} := \sigma\{\tilde{N}(\triangle) :\ \triangle \in \mathcal{B}(\mathbb{R}_0 \times (0, t)), 0 \leq t \leq T\} \vee \mathcal{N}$$

are the augmented filtrations generated by $B(\cdot)$ and $\tilde{N}(\cdot, \cdot)$ respectively. We denote by $\mathcal{F}_t = \mathcal{F}_t^B \otimes \mathcal{F}_t^{\tilde{N}}, 0 \leq t \leq T$ the augmented filtration generated by B and $\tilde{N}$. Let $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ be the filtration such that

$$\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{F} \quad \forall t\ \in [0, T],$$

where $T > 0$ is a fixed terminal time.

Let $\varphi(t, \omega)$ be a $\mathbb{G}$-adapted process. Then

$$\int_0^T \varphi(t, \omega)dB(t) \tag{2.1}$$

makes no sense in the normal settings. Similarly, for a $\mathbb{G}$-adapted process $\psi(t, z, \omega)$,

$$\int_0^T \int_{\mathbb{R}_0} \psi(t, z, \omega)\tilde{N}(dt, dz) \tag{2.2}$$

does not make sense either. Therefore, it is natural to use forward integrals to handle this problem and to make the integrals (2.1) and (2.2) well defined.

Definition 2.1. Let $\varphi(\cdot,\omega)$, $\omega \in \Omega_B$ be a measurable process. The forward integral of φ with respect to Brownian motion is defined by

$$\int_0^\infty \varphi(t,\omega)d^-B(t) = \lim_{\varepsilon\to 0}\int_0^\infty \varphi(t,\omega)\frac{B(t+\varepsilon)-B(t)}{\varepsilon}\,dt$$

if the limit exists in probability. Then φ is called *forward integrable* with respect to Brownian motion. If the limit exists also in $L^2(P)$, we write $\varphi \in \mathbb{D}^B$.

In particular, we recall the following result.

Lemma 2.2. *Let φ be forward integrable and càglàd (i.e., left continuous with right limits). Then for any partition $0 = t_0 < t_1 < \ldots < t_N = T$*

$$\int_0^T \varphi(t,\omega)d^-B(t) = \lim_{|\triangle t|\to 0}\sum_j \varphi(t_j)\triangle B(t_j),$$

where $\triangle B(t_j) = B(t_{j+1}) - B(t_j)$ and $|\triangle t| = \sup_{j=0,\ldots,N-1}\triangle t_j$.

Remark 2.3. Let $\varphi \in \mathbb{D}^B$ be a càglàd process. If $B(\cdot)$ is a $\mathbb{G}$-semimartingale, then $\int_0^T \varphi(t,\omega)dB(t)$ exists as a semimartingale integral and

$$\int_0^T \varphi(t,\omega)d^-B(t) := \int_0^T \varphi(t,\omega)dB(t).$$

Let us now give the corresponding definition of forward integral with respect to the compensated Poisson random measure.

Definition 2.4. Let $\varphi(\cdot,z,\omega)$, $z \in \mathbb{R}_0$, $\omega \in \Omega_\eta$ be a measurable random field. The forward integral of φ with respect to the compensated Poisson random measure is defined by

$$\int_0^\infty\int_{\mathbb{R}_0} \varphi(t,z)\tilde{N}(d^-t,dz) = \lim_{n\to\infty}\int_0^\infty\int_{U_n} \varphi(t,z)\,\tilde{N}(dt,dz)$$

if the limit exists in probability. Here, U_n is an increasing sequence of compact sets where $U_n \subseteq \mathbb{R}_0$, $\nu_{\mathcal{F}}(U_n) < \infty$ and $\bigcup_{n=1}^\infty U_n = \mathbb{R}_0$. Then, φ is called *forward integrable* with respect to the Poisson random measure. If the limit exists in $L^2(P)$, we write $\varphi \in \mathbb{D}^{\tilde{N}}$.

Remark 2.5. Let $\varphi \in \mathbb{D}^{\tilde{N}}$ be càglàd. If $\int_0^\cdot\int_{\mathbb{R}_0} \varphi(t,z,\omega)\tilde{N}(dt,dz)$ is a $\mathbb{G}$-semimartingale, then

$$\int_0^T\int_{\mathbb{R}_0} \varphi(t,z,\omega)\tilde{N}(d^-t,dz) := \int_0^T\int_{\mathbb{R}_0} \varphi(t,z,\omega)\tilde{N}(dt,dz),\ \ a.s..$$

The last result we recall in this section is the Itô formula for forward integrals. We first define the forward process.

Definition 2.6. A *forward process* is a measurable stochastic function $X(\cdot)$, that admits the representation

$$X(t) = X(0) + \int_0^t \alpha(s)ds + \int_0^t \beta(s)d^-B(s) + \int_0^t\int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(d^-s,dz),\ \forall t \in [0,T],$$

where $\int_0^T (|\alpha(s)| + \beta(s)^2)ds < \infty$, almost surely, $\gamma(t,z)$ is continuous in z around zero for a.a. $t \in [0,T]$ such that

$$\int_0^t \int_{\mathbb{R}_0} |\gamma(s,z)|^2 \, \nu_{\mathcal{F}}(ds,dz) < \infty, \quad \text{almost surely for } t \in [0,T].$$

Moreover, $\beta(\cdot)$ and $\gamma(\cdot,\cdot)$ are forward integrable with respect to Brownian motion and compensated Poisson random measure. A shorthand notation for this is

$$d^-X(t) = \alpha(t)dt + \beta(t)d^-B(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(d^-t,dz). \tag{2.3}$$

Theorem 2.7. ***(Itô formula for forward integrals).***
Let $X(\cdot)$ be a forward process of the form (2.3) and define $Y(t) = f(X(t))$ for any $f \in \mathbb{C}^2(\mathbb{R})$ and all $t \in [0,T]$. Then $Y(\cdot)$ is also a forward process and

$$\begin{aligned} d^-Y(t) = &\Big[f'(X(t))\alpha(t) + \frac{1}{2}f''(X(t))\beta(t)^2 + \int_{\mathbb{R}_0} \Big\{f(X(t^-) + \gamma(t,z)) \\ &- f(X(t^-)) - f'(X(t^-))\gamma(t,z)\Big\}\nu_{\mathcal{F}}(dz)\Big]dt + f'(X(t))\beta(t)d^-B(t) \\ &+ \int_{\mathbb{R}_0} \big(f(X(t^-) + \gamma(t,z)) - f(X(t^-))\big)\tilde{N}(d^-t,dz), \end{aligned}$$

where f' and f'' are the first and second derivatives respectively.

Proof. We refer to Russo and Valois [28] for the proof in the Brownian motion case and to Di Nunno *et al.* [6] for the pure jump Lévy process case. □

3. The Main Problem

Assume there is a riskless and a risky asset in an arbitrage-free financial market. The price per unit of the riskless asset is denoted by $S_0(\cdot)$ and satisfies the following ordinary differential equation (O.D.E.)

$$\begin{aligned} dS_0(t) &= r(t)S_0(t)dt, \\ S_0(0) &= 1. \end{aligned}$$

The risky asset has a price process $S_1(\cdot)$ defined by

$$\begin{aligned} dS_1(t) &= S_1(t^-)[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz)], \\ S_1(0) &> 0. \end{aligned}$$

Assume that the coefficients $r(t) = r(t,\omega), \mu(t) = \mu(t,\omega), \sigma(t) = \sigma(t,\omega)$, and $\gamma(t,z) = \gamma(t,z,\omega)$ satisfy the following conditions:

(1) $r(\cdot), \mu(\cdot), \sigma(\cdot), \gamma(\cdot,z)$ are $\mathbb{F}$-adapted càglàd processes.
(2) $\gamma(t,z) > -1 \quad dt \times \nu_{\mathcal{F}}(dz)$-a.e.
(3) $\int_0^T \{|r(t)| + |\mu(t)| + \sigma(t)^2 + \int_{\mathbb{R}_0} \gamma(t,z)^2 \, \nu_{\mathcal{F}}(dz)\}dt < \infty$ a.s.

In this paper, we will consider an agent who wants to maximize his expected intertemporal utility of consumption and terminal value of wealth when the portfolio is allowed to anticipate the future. Hence, we assume that the informed agent's portfolio and consumption choices are adapted to the larger filtration $\mathbb{G}$. Note that $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ can not be chosen freely. For example, if $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(X)$, where X is any hedgeable $\mathcal{F}_T$-measurable random variable, then the informed agent immediately obtain the arbitrage opportunity. For further information, see [8], [9], [11] and [17]. In this and the next section, we define the stochastic control problem and characterize the controls under the filtration $\mathbb{G}$ such that the additional information for an insider does not blow up the value of the problem.

Let $\pi(t)$ be the fraction of the wealth invested in the stock (risky asset) at time t by an insider. Since $\pi(\cdot)$ is a $\mathbb{G}$-adapted process, it is natural to use the forward integration to make the integrals well defined. The corresponding wealth process $X^{(c,\pi)}(\cdot)$ of the insider is given by

$$\begin{aligned} d^-X^{(c,\pi)}(t) &= X^{(c,\pi)}(t^-)[\{r(t)+(\mu(t)-r(t))\pi(t)\}dt+\sigma(t)\pi(t)d^-B(t) \\ &\quad +\pi(t)\int_{\mathbb{R}_0}\gamma(t,z)\tilde{N}(d^-t,dz)]-c(t)dt \end{aligned}$$

with initial value $X^{(c,\pi)}(0)=x$.

We assume that the agent has a logarithmic utility function. It is convenient to use such functions because it has *iso-elastic marginal utility* which means that an agent has the same relative risk-tolerance as toward the end of his life. Moreover, as $X(t)\geq 0$ for all $t\in[0,T]$, we can assume that the insider has a relative consumption rate $\lambda(\cdot)$ defined by

$$\lambda(t) := \frac{c(t)}{X^{(c,\pi)}(t)}$$

without loss of generality. If $X^{(c,\pi)}(T)=0$, we put $\lambda(T)=0$. Then the corresponding wealth dynamic of the insider in terms of relative consumption rate λ and portfolio π can be rewritten by the following forward S.D.E.

$$\begin{aligned} d^-X^{(\lambda,\pi)}(t) &= X^{(\lambda,\pi)}(t^-)[\{r(t)-\lambda(t)+(\mu(t)-r(t))\pi(t)\}dt \\ &\quad +\sigma(t)\pi(t)d^-B(t)+\pi(t)\int_{\mathbb{R}_0}\gamma(t,z)\tilde{N}(d^-t,dz)], \end{aligned} \tag{3.1}$$

for all $t\in[0,T]$, where the initial wealth is $X^{(\lambda,\pi)}(0)=x>0$.

Problem 3.1. Find the optimal relative consumption rate $\lambda^*(\cdot)$ and the optimal portfolio $\pi^*(\cdot)$ for an insider subject to his budget constraint, i.e. find the pair $(\lambda^*,\pi^*)\in\mathcal{A}$ which maximizes the performance function given by

$$J(\lambda^*,\pi^*) := \sup_{(\lambda,\pi)\in\mathcal{A}}\mathbb{E}\left[\int_0^T e^{-\delta(t)}\ln(\lambda(t)X^{(\lambda,\pi)}(t))dt+Ke^{-\delta(T)}\ln X^{(\lambda,\pi)}(T)\right].$$

Here, $\delta(t)\geq 0$ is the discount rate for all $t\in[0,T]$. It is a given bounded deterministic function satisfying $\int_0^T e^{-\delta(u)}du+Ke^{-\delta(T)}\neq 0$. $T>0$ is a fixed terminal time and $X^{(\lambda,\pi)}(\cdot)$ is the wealth process satisfying the equation (3.1).

$\mathcal{A}$ is the set of all admissible control pairs which will be defined in the following definition.

Definition 3.2. A $\mathbb{G}$-adapted stochastic process pair (λ, π) is called *admissible* if

(i) $\int_0^T \lambda(s)ds < \infty$ a.s..
(ii) $\pi(t)$, $t \in [0,T]$ is càglàd.
(iii) $\pi(\cdot)\sigma(\cdot)$ and $\pi(\cdot)\gamma(\cdot,z)$ are forward integrable with respect to Brownian motion and the compensated Poisson random measure, respectively.
(iv) $1 + \pi(t)\gamma(t,z) > \varepsilon_\pi$ for a.a. (t,z) with respect to $dt \times \nu_{\mathcal{F}}(dz)$, for some $\varepsilon_\pi \in (0,1)$ depending on π.
(v) $\int_0^T \left\{ |(\mu(s) - r(s))\pi(s)| + \sigma^2(s)\pi(s)^2 + \int_{\mathbb{R}_0} \pi(s)^2\gamma(s,z)^2\nu_{\mathcal{F}}(dz) \right\} ds < \infty$ almost surely.
(vi) $\mathbb{E}\left[\int_0^T e^{-\delta(s)} |\ln \lambda(s)| ds + Ke^{-\delta(T)} |\ln X^{(\lambda,\pi)}(T)| \right] < \infty$.

4. Characterization of the Optimal Consumption and Investment Choice

Since we use an iso-elastic utility function, this optimal consumption rate depends only on the discount rate. The other parameters in the economy such as interest rates or volatility do not appear. However, the consumption of the agent is depending on these coefficients through the wealth. The next step is to show that the optimal consumption rate found in Øksendal [25] is also optimal for Problem 3.1, i.e., the relative consumption rate can be chosen independently from the optimal portfolio if the terminal wealth is added to the problem when the agent has logarithmic utility function. This result for non-anticipative information was initially established by Samuelson [30].

Theorem 4.1. *Define* $\hat{\lambda}$ *as*

$$\hat{\lambda}(t) := \frac{e^{-\delta(t)}}{\int_t^T e^{-\delta(s)}ds + K\, e^{-\delta(T)}}; \quad t \geq 0. \tag{4.1}$$

Then $\hat{\lambda}$ *is an* optimal relative consumption rate *independent of the portfolio chosen, in the sense that* $(\hat{\lambda}, \pi) \in \mathcal{A}$ *and*

$$J(\hat{\lambda}, \pi) \geq J(\lambda, \pi)$$

for all λ *and* π *such that* $(\lambda, \pi) \in \mathcal{A}$.

Proof. The proof can be shown using the additive separability of log-utility function. □

Note that since the optimal relative consumption rate, $\lambda^* = \hat{\lambda}$ does not depend on the portfolio choice, the main problem turns to :

Problem 4.2. Find π^* such that $(\lambda^*, \pi^*) \in \mathcal{A}$ and

$$J(\lambda^*, \pi^*) := \tilde{J}(\pi^*) = \sup_{(\lambda^*,\pi)\in\mathcal{A}} \mathbb{E}\left[\int_0^T e^{-\delta(t)} \ln(\lambda^*(t) X^{(\lambda^*,\pi)}(t))dt + Ke^{-\delta(T)} \ln X^{(\lambda^*,\pi)}(T)\right].$$

For all $(\lambda, \pi) \in \mathcal{A}$, let us define M_π and $Y_\pi(t)$ as follows:

$$\begin{aligned} M_\pi(t) \quad := \quad & \int_0^t \left\{\mu(s) - r(s) - \sigma(s)^2\pi(s) - \int_{\mathbb{R}_0} \frac{\pi(s)\gamma(s,z)^2}{1+\pi(s)\gamma(s,z)}\nu_{\mathcal{F}}(dz)\right\} ds \\ & + \int_0^t \sigma(s) d^- B(s) + \int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)} \tilde{N}(d^- s, dz) \end{aligned} \tag{4.2}$$

and

$$Y_\pi(t) := \int_0^t e^{-\delta(s)} M_\pi(s) ds + M_\pi(t) \Big(\int_t^T e^{-\delta(s)} ds + Ke^{-\delta(T)} \Big), \tag{4.3}$$

for all $t \in [0, T]$. Moreover, we make the following assumptions:

Assumption 4.3.

(A.1) $\forall (\lambda, \pi), (\lambda, \beta) \in \mathcal{A}$ *with* β *bounded, there exists positive* τ *such that the family* $\{\mid M_{\pi+\varepsilon\beta}(T) \mid\}_{0\le\varepsilon\le\tau}$ *is uniformly integrable.*

(A.2) *For all* $t \in [0, T]$ *the process pair* (λ, π) *where* $\pi(s) := \chi_{(t,t+h]}(s)\beta_0(\omega)$*, with* $h > 0$ *and* $\beta_0(\omega)$ *being a bounded* $\mathcal{G}_t$*-measurable random variable, belongs to* $\mathcal{A}$*.*

The following theorem plays a crucial role to obtain the necessary conditions for the optimal consumption and investment choice.

Theorem 4.4. *Suppose* $(\lambda^*, \pi^*) \in \mathcal{A}$ *is optimal for Problem 3.1. Then* $Y_{\pi^*}(\cdot)$ *defined in* (4.3) *is a* $\mathbb{G}$*-martingale.*

Proof. Suppose that $(\lambda^*, \pi^*) \in \mathcal{A}$ is optimal for the insider. We can choose $\beta(\cdot)$ such that $(\lambda^*, \beta) \in \mathcal{A}$. Then $(\lambda^*, \pi^*(\cdot) + y\beta(\cdot)) \in \mathcal{A}$, for all y small enough. Since $J(\lambda^*, \pi^* + y\beta)$ also denoted as $\tilde{J}(\pi^* + y\beta)$ is maximal at π^*, we have

$$\frac{d}{dy}\tilde{J}(\pi^* + y\beta)|_{y=0} = 0,$$

which implies

$$\begin{aligned} \mathbb{E}\Bigg[\int_0^T e^{-\delta(s)} \Bigg(\int_0^s \beta(u)\{\mu(u) - r(u) - \pi^*(u)\sigma(u)^2 \\ - \int_{\mathbb{R}_0} (\gamma(u,z) - \frac{\gamma(u,z)}{1+\pi^*(u)\gamma(u,z)})\nu_{\mathcal{F}}(dz)\}du + \int_0^s \beta(u)\sigma(u)d^- B(u) \\ + \int_0^s \int_{\mathbb{R}_0} \frac{\beta(u)\gamma(u,z)}{1+\pi^*(u)\gamma(u,z)} \tilde{N}(d^- u, dz)\Bigg) ds \end{aligned} \tag{4.4}$$

$$+Ke^{-\delta(T)}\left(\int_0^T \beta(u)\sigma(u)d^-B(u) + \int_0^T\int_{\mathbb{R}_0}\frac{\beta(u)\gamma(u,z)}{1+\pi^*(u)\gamma(u,z)}\tilde{N}(d^-u,dz)\right.$$
$$+\int_0^T \beta(u)\{\mu(u)-r(u)-\pi^*(u)\sigma(u)^2$$
$$\left.-\int_{\mathbb{R}_0}(\gamma(u,z)-\frac{\gamma(u,z)}{1+\pi^*(u)\gamma(u,z)})\nu_{\mathcal{F}}(dz)\}du\right]=0. \tag{4.5}$$

Let us fix $t\in[0,T)$ and $h>0$ such that $t+h\le T$. We can choose β of the form

$$\beta(s) := \chi_{(t,t+h]}(s)\beta_0 \tag{4.6}$$

where β_0 is a bounded $\mathcal{G}_t$-measurable random variable. Rewriting equation (4.5) by using the equation (4.6), we obtain :

$$\mathbb{E}\left[\beta_0\int_t^{t+h} e^{-\delta(s)}\left(\int_t^s\{\mu(u)-r(u)-\pi^*(u)\sigma(u)^2-\int_{\mathbb{R}_0}\frac{\pi^*(u)\gamma(u,z)^2}{1+\pi^*(u)\gamma(u,z)}\nu_{\mathcal{F}}(dz)\}du\right.\right.$$
$$+\int_t^s\sigma(u)d^-B(u)+\int_t^s\int_{\mathbb{R}_0}\frac{\gamma(u,z)}{1+\pi^*(u)\gamma(u,z)}\tilde{N}(d^-u,dz)\Big)ds$$
$$+\beta_0\int_{t+h}^T e^{-\delta(s)}\left(\int_t^{t+h}\sigma(u)d^-B(u)+\int_t^{t+h}\int_{\mathbb{R}_0}\frac{\gamma(u,z)}{1+\pi^*(u)\gamma(u,z)}\tilde{N}(d^-u,dz)\right.$$
$$\left.\left.+\int_t^{t+h}\Big\{\mu(u)-r(u)-\pi^*(u)\sigma(u)^2-\int_{\mathbb{R}_0}\frac{\pi^*(u)\gamma(u,z)^2}{1+\pi^*(u)\gamma(u,z)}\nu_{\mathcal{F}}(dz)\Big\}du\right)ds\right]$$
$$+\mathbb{E}\left[Ke^{-\delta(T)}\beta_0\left(\int_t^{t+h}\Big\{\mu(u)-r(u)-\pi^*(u)\sigma(u)^2-\int_{\mathbb{R}_0}\frac{\pi^*(u)\gamma(u,z)^2}{1+\pi^*(u)\gamma(u,z)}\nu_{\mathcal{F}}(dz)\Big\}du\right.\right.$$
$$\left.\left.+\int_t^{t+h}\sigma(u)d^-B(u)+\int_t^{t+h}\int_{\mathbb{R}_0}\frac{\gamma(u,z)}{1+\gamma(u,z)\pi^*(u)}\tilde{N}(d^-u,dz)\right)\right]=0.$$

Let us define $M_\pi(\cdot)$ as in equation (4.2), then the above equation turns to :

$$\mathbb{E}\left[\beta_0\left(\int_t^{t+h}e^{-\delta(s)}M_{\pi^*}(s)ds+M_{\pi^*}(t+h)\left(\int_{t+h}^T e^{-\delta(s)}ds+Ke^{-\delta(T)}\right)\right.\right.$$
$$\left.\left.-M_{\pi^*}(t)\left(\int_t^T e^{-\delta(s)}ds+Ke^{-\delta(T)}\right)\right)\right]=0.$$

Let us define

$$N_{\pi^*}(t) := \int_0^t e^{-\delta(s)}M_{\pi^*}(s)ds$$

and

$$P_{\pi^*}(t) := M_{\pi^*}(t)\left(\int_t^T e^{-\delta(s)}ds+Ke^{-\delta(T)}\right).$$

Hence, we have

$$\mathbb{E}\left[\beta_0\left(N_{\pi^*}(t+h)-N_{\pi^*}(t)+P_{\pi^*}(t+h)-P_{\pi^*}(t)\mid\mathcal{G}_t\right)\right]=0.$$

By using the equation (4.3), we get

$$\mathbb{E}[\beta_0(Y_{\pi^*}(t+h) - Y_{\pi^*}(t))] = 0.$$

Since this holds for all bounded $\mathcal{G}_t$-measurable β_0, we have :

$$\mathbb{E}[Y_{\pi^*}(t+h)|\mathcal{G}_t] = Y_{\pi^*}(t).$$

Hence Y_{π^*} is a $\mathbb{G}$-martingale. □

The initial enlargement of filtration technique is a common methodology used in stochastic control problems with anticipative information. The main assumption in this method is that the conditional distribution of the random time is absolutely continuous to a measure. It implies that every $\mathbb{F}$-martingale is a semimartingale with respect to the enlarged filtration. In this paper, we do not assume this strict condition and indeed we have this as a result by using Theorem 4.4 and forward integral techniques. In the following theorem, we show that if there exists admissible optimal portfolio and consumption choices, then $\mathbb{F}$-Brownian motion is a $\mathbb{G}$-semimartingale.

Theorem 4.5. *Suppose $\gamma(t,z) \neq 0$ and $\sigma(t) \neq 0$ for a.a. (t,z,ω). Suppose that there exist optimal relative consumption rate and optimal portfolio $(\lambda^*, \pi^*) \in \mathcal{A}$ for Problem 3.1. Then*

(i) *$B(\cdot)$ is a $(\mathbb{G}, P)$-semimartingale. Therefore, there exists an adapted finite variation process $\alpha(\cdot)$ such that the process $\hat{B}(\cdot)$ defined as*

$$\hat{B}(t) = B(t) - \int_0^t \alpha(s)ds; \qquad \forall t \in [0,T]$$

is a $\mathbb{G}$-Brownian motion.

(ii) *The process*

$$\int_0^{\cdot} \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)} \tilde{N}(d^- s, dz)$$

is a $\mathbb{G}$-semimartingale.

(iii) *The process*

$$\int_0^{\cdot} \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds,dz)$$

is a $\mathbb{G}$-semimartingale.

(iv) *The optimal portfolio π satisfies the following equation:*

$$\int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)} \left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)} \right) (\nu_{\mathcal{G}} - \nu_{\mathcal{F}})(ds,dz)$$
$$+ \int_0^t \Bigg\{ \left(\mu(s) - r(s) - \sigma(s)^2\pi(s) + \sigma(s)\alpha(s) - \int_{\mathbb{R}_0} \frac{\pi(s)\gamma(s,z)^2}{1+\pi(s)\gamma(s,z)} \nu_{\mathcal{F}}(dz) \right)$$
$$\times \left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)} \right) \Bigg\} ds = 0, \quad \forall t \in [0,T].$$

(v) *The optimal relative consumption rate is given by*

$$\lambda^*(t) = \frac{e^{-\delta(t)}}{\int_t^T e^{-\delta(s)}ds + K\, e^{-\delta(T)}}; \quad \forall t \in [0,T].$$

Proof. Let $(\lambda, \pi) \in \mathcal{A}$ be an optimal control choice for the Problem 3.1. By applying the Fubini's Theorem to $Y_\pi(t)$ in the equation (4.3), we can write:

$$\begin{aligned} Y_\pi(t) = &\int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)} \left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)} \right) \tilde{N}(d^-s, dz) \\ &+ \int_0^t \sigma(s) \left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)} \right) d^-B(s) \\ &+ \int_0^t \Big\{ \left(\mu(s) - r(s) - \sigma(s)^2\pi(s) - \int_{\mathbb{R}_0} \frac{\pi(s)\gamma(s,z)^2}{1+\pi(s)\gamma(s,z)} \nu_{\mathcal{F}}(dz) \right) \\ &\times \left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)} \right) \Big\} ds. \end{aligned}$$

Note that since $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \in [0,T]$, the Poisson random measure $N(dt,dz)$ has a unique compensator with respect to the enlarged filtration $\mathbb{G}$, say $\nu_{\mathcal{G}}(dt,dz)$ for all $(t,z) \in [0,T] \times \mathbb{R}_0$. Using the orthogonal decomposition into a continuous part $Y_\pi^c(t)$ and a discontinuous part $Y_\pi^d(t)$, we get as follows

$$\begin{aligned} Y_\pi^c(t) = &\int_0^t \sigma(s) \left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)} \right) d^-B(s) \\ &- \int_0^t \sigma(s)\alpha(s) \left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)} \right) ds \\ Y_\pi^d(t) = &\int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)} \left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)} \right) \tilde{N}(d^-s, dz) \\ &+ \int_0^t \int_{\mathbb{R}_0} \theta(s,z)(\nu_{\mathcal{F}} - \nu_{\mathcal{G}})(ds,dz), \end{aligned}$$

where $\alpha(s)$ and $\theta(s,\cdot)$ are $\mathcal{G}_s$-measurable processes for all $s \in [0,T]$ such that

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}_0} \theta(s,z)(\nu_{\mathcal{F}} - \nu_{\mathcal{G}})(ds,dz) - \int_0^t \sigma(s)\alpha(s) \left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)} \right) ds \\ &= \int_0^t \Big\{ \left(\mu(s) - r(s) - \sigma(s)^2\pi(s) - \int_{\mathbb{R}_0} \frac{\pi(s)\gamma(s,z)^2}{1+\pi(s)\gamma(s,z)} \nu_{\mathcal{F}}(dz) \right) \\ &\quad \times \left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)} \right) \Big\} ds. \end{aligned}$$

(i) For the continuous part $Y_\pi^c(t)$, we use the fact that

$$\int_0^t \frac{1}{\sigma(s)\Big(\int_0^T e^{-\delta(u)}du + Ke^{-\delta(T)} \Big)} dY_\pi^c(s) = B(t) - \int_0^t \alpha(s)ds, \ t \in [0,T]$$

is a $\mathcal{G}$-martingale. Then we obtain directly that $B(\cdot)$ is a $\mathcal{G}$-semimartingale.

(ii) Since $Y_\pi(\cdot)$ is a $\mathcal{G}$-martingale, we can easily show that $\Gamma(\cdot)$ defined as

$$\Gamma(t) := \int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)} \Big(\int_s^T e^{-\delta(u)} du + Ke^{-\delta(T)} \Big) \tilde{N}(d^- s, dz) + \int_0^t \sigma(s) \Big(\int_s^T e^{-\delta(u)} du + Ke^{-\delta(T)} \Big) d^- B(s), \ t \in [0,T]$$

is a $\mathcal{G}$-semimartingale. Then,

$$\int_0^\cdot \Big(\int_s^T e^{-\delta(u)} du + Ke^{-\delta(T)} \Big)^{-1} d\Gamma(s) = \int_0^\cdot \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)} \tilde{N}(d^- s, dz) + \int_0^\cdot \sigma(s) d^- B(s)$$

is also a $\mathcal{G}$-semimartingale. Finally, using (i) we conclude that

$$\int_0^\cdot \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)} \tilde{N}(d^- s, dz)$$

is a $\mathcal{G}$-semimartingale.

(iii) By (ii) and Remark 2.5, we know that

$$\int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)} (\nu_{\mathcal{F}} - \nu_{\mathcal{G}})(ds, dz), \ t \in [0,T]$$

is of finite variation. Using Hypothesis (iv) of Definition 3.2, it follows that

$$\int_0^t \int_{\mathbb{R}_0} \gamma(s,z)(\nu_{\mathcal{F}} - \nu_{\mathcal{G}})(ds, dz), \ t \in [0,T]$$

is of finite variation. Note that the $\mathcal{G}$-martingale

$$\int_0^\cdot \int_{\mathbb{R}_0} \gamma(s,z)(N - \nu_{\mathcal{G}})(ds, dz)$$

can be written as :

$$\int_0^t \int_{\mathbb{R}_0} \gamma(s,z)(N - \nu_{\mathcal{G}})(ds, dz) = \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds, dz) + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)(\nu_{\mathcal{F}} - \nu_{\mathcal{G}})(ds, dz), \ t \in [0,T]$$

and since $\int_0 \int_{\mathbb{R}_0} \gamma(s,z)(\nu_{\mathcal{F}} - \nu_{\mathcal{G}})(ds, dz)$ is of finite variation then

$$\int_0^\cdot \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds, dz)$$

is a $\mathcal{G}$-semimartingale.

(iv) Let us rewrite $Y_\pi(t)$ as

$$\begin{aligned}
Y_\pi(t) &= \int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)}\Big(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\Big)(N-\nu_{\mathcal{G}})(ds,dz) \\
&+ \int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)}\Big(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\Big)(\nu_{\mathcal{G}}-\nu_{\mathcal{F}})(ds,dz) \\
&+ \int_0^t \Big\{ \Big(\mu(s) - r(s) - \sigma(s)^2\pi(s) + \sigma(s)\alpha(s) - \int_{\mathbb{R}_0} \frac{\pi(s)\gamma(s,z)^2}{1+\pi(s)\gamma(s,z)}\nu_{\mathcal{F}}(dz)\Big) \\
&\times \Big(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\Big)\Big\}ds \\
&+ \int_0^t \sigma(s)\Big(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\Big)d\hat{B}(s).
\end{aligned}$$

Hence by the martingale representation theorem, we conclude that the finite variation part is zero, i.e.,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)}\Big(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\Big)(\nu_{\mathcal{G}}-\nu_{\mathcal{F}})(ds,dz) \\
&+ \int_0^t \Big\{\mu(s) - r(s) - \sigma(s)^2\pi(s) + \sigma(s)\alpha(s) - \int_{\mathbb{R}_0} \frac{\pi(s)\gamma(s,z)^2}{1+\pi(s)\gamma(s,z)}\nu_{\mathcal{F}}(dz)\Big\} \\
&\times \Big(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\Big)ds = 0.
\end{aligned}$$

□

Finally, as a Corollary, let us present the results for an uninformed agent:

Corollary 4.6. *Suppose $\mathcal{F}_t = \mathcal{G}_t$, for all $t \in [0,T]$. Then the optimal portfolio $\pi(\cdot)$ solves the following equation:*

$$\mu(t) - r(t) - \sigma(t)^2\pi(t) - \int_{\mathbb{R}_0} \frac{\pi(t)\gamma(t,z)^2}{1+\pi(t)\gamma(t,z)}\nu_{\mathcal{F}}(dz) = 0, \quad \forall t \in [0,T]$$

and the optimal relative consumption rate $\lambda(t)$ is given by

$$\lambda(t) = \frac{e^{-\delta(t)}}{\int_t^T e^{-\delta(u)}du + K\,e^{-\delta(T)}}; \quad \forall t \in [0,T].$$

Proof. These results can be directly derived from Theorem 4.5. □

5. Examples

In this section, we give two examples to illustrate our results. These examples were already treated in other papers for the optimal portfolio choices. The aim of this section is to show that our approach is coherent and give the same results as the ones obtained using enlargement of filtration theory.

Example 5.1. The Brownian motion case.

Suppose that $\gamma(t,z)=0$ and $\sigma(t)\neq 0$ for almost all (t,z). We denote by $\pi_i^*(t)$ and $\pi_h^*(t)$ the optimal portfolios for the insider and the uninformed (honest) agent, respectively. By Theorem 4.5, the optimal portfolio $\pi_i^*(t)$ satisfies the following equation

$$\int_0^t \left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right)\left\{\mu(s)-r(s)-\sigma(s)^2\pi_i^*(s)+\sigma(s)\alpha(s)\right\}ds = 0,$$

for all $t\in[0,T]$. Then, we obtain an explicit solution for $\pi_i^*(t)$:

$$\pi_i^*(t) = \frac{\mu(t)-r(t)}{\sigma(t)^2} + \frac{\alpha(t)}{\sigma(t)}, \quad \forall t\in[0,T]$$

and the optimal relative consumption rate for the insider $\lambda_i^*(t)$ given by :

$$\lambda_i^*(t) = \frac{e^{-\delta(t)}}{\int_t^T e^{-\delta(s)}ds + Ke^{-\delta(T)}}.$$

For the uninformed agent, by Corollary 4.6, $\pi_h^*(t)$ and $\lambda_h^*(t)$ are given by :

$$\pi_h^*(t) = \frac{\mu(t)-r(t)}{\sigma(t)^2}, \qquad \lambda_h^*(t) = \frac{e^{-\delta(t)}}{\int_t^T e^{-\delta(s)}ds + K\,e^{-\delta(T)}} = \lambda_i^*(t).$$

By (i) in Theorem 4.5, $B(t)$ is a $\mathcal{G}_t$-semimartingale then

$$X_i^{\left(c_{\lambda_i^*},\pi_i^*\right)}(t) = X_h^{\left(c_{\lambda_h^*},\pi_h^*\right)}(t)\exp\left\{\frac{1}{2}\int_0^t \alpha(s)^2ds + \int_0^t \alpha(s)d\hat{B}(s)\right\} \tag{5.1}$$

where $X_i^{\left(c_{\lambda_i^*},\pi_i^*\right)}(t)$ and $X_h^{\left(c_{\lambda_h^*},\pi_h^*\right)}(t)$ are the optimal wealth processes for the insider and the uninformed agent respectively.
Hence,

$$\begin{aligned} J_i\left(c_{\lambda_i^*},\pi_i^*\right) = J_h\left(c_{\lambda_h^*},\pi_h^*\right) &+ \frac{1}{2}\mathbb{E}\left[\int_0^T e^{-\delta(t)}\int_0^t \alpha(s)^2ds\,dt\right] \\ &+ \frac{1}{2}K\mathbb{E}\left[e^{-\delta(T)}\int_0^T \alpha(s)^2ds\right]. \end{aligned} \tag{5.2}$$

Remark 5.2. Note that although the optimal relative consumption rates are the same, the optimal consumption rates are not the same among the informed and uninformed agent by equation (5.1).

Proposition 5.3. *Let $\mathcal{G}_t = \mathcal{F}_t^B \vee \sigma(B(T_0)), T_0 > T$ for all $t\in[0,T]$. Assume that there exist optimal control choices for the Problem 3.1. Let $\delta(t)$ and $\gamma(t,z)$ are equal to zero for all $(t,z)\in[0,T]\times\mathbb{R}_0$. Then the additional performance function of an insider is equal to*

$$\frac{1}{2}\left[(T_0-T)\ln(T_0-T)+T\right] + \frac{K}{2}\ln\left(\frac{T_0}{T_0-T}\right).$$

Proof. If we restrict the enlarged filtration to be $\mathcal{G}_t = \mathcal{F}_t^B \vee \sigma(B(T_0)), T_0 > T$, then

$$\alpha(t) = \frac{B(T_0) - B(t)}{T_0 - t}.$$

and by the equation (11), the performance function of the informed agent can be written in terms of the performance function of the uninformed one as follows :

$$\begin{aligned} J_i\left(c_{\lambda_i^*}, \pi_i^*\right) &= J_h\left(c_{\lambda_h^*}, \pi_h^*\right) + \frac{1}{2}\int_0^T \int_0^t \frac{1}{T_0 - s} ds + \frac{K}{2}\int_0^T \frac{1}{T_0 - s} ds \\ &= J_h\left(c_{\lambda_h^*}, \pi_h^*\right) + \frac{1}{2}\Big[(T_0 - T)\ln(T_0 - T) + T\Big] + \frac{K}{2}\ln\left(\frac{T_0}{T_0 - T}\right). \end{aligned}$$

□

Example 5.4. The mixed case.

Suppose that $\gamma(t, z) = z$ and $\sigma(t) \neq 0$ for almost all (t, z). We consider the enlarged filtration $\mathcal{G}_t' = \mathcal{F}_t \vee \sigma(B(T_0), \eta(T_0)), T_0 > T$ and take the following assumptions:

(1) The informed agent has access to the filtration $\mathcal{G}_t$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{G}_t'$, $t \in [0, T]$.
(2) The Lévy measure $\nu_{\mathcal{F}}$ is given by $\nu_{\mathcal{F}}(ds, dz) = \rho\delta_1(dz)ds$ where $\delta_1(dz)$ is the unit point mass at 1.
(3) $\eta(t)$ is defined as $\eta(t) = Q(t) - \rho t$ with Q being a Poisson process of intensity ρ.

Using the results of Di Nunno *et al.* [7] Section 5, we obtain the following optimal portfolio $\pi_i^*(t)$:

$$\pi_i^*(t) = \pi_h^*(t) + \frac{\zeta(t)}{\sigma(t)}$$

with

$$\begin{aligned} \pi_h^*(t) &= \frac{\mu(t) - r(t)}{\sigma(t)^2} - \frac{\rho}{\sigma(t)^2}, \\ \zeta(t) &= \frac{1}{2\sigma(t)}\Big[-\mu(t) + r(t) + \rho + \sigma(t)\alpha(t) - \sigma(t)^2 \\ &\qquad + \sqrt{\left(\mu(t) - r(t) - \rho + \sigma(t)\alpha(t) + \sigma(t)^2\right)^2 + 4\sigma(t)^2\theta(t)}\Big], \\ \alpha(t) &= \frac{\mathbb{E}[B(T_0) - B(s)|\mathcal{G}_s]^-}{T_0 - s}, \\ \theta(t) &= \frac{\mathbb{E}[Q(T_0) - Q(s)|\mathcal{G}_s]^-}{T_0 - s}, \end{aligned}$$

where the notation $\mathbb{E}[...]^-$ denotes the left limit in s.

Moreover we have the optimal consumption rates $\lambda_i^*(t)$ and $\lambda_h^*(t)$ for the informed and uninformed agents respectively:

$$\lambda_i^*(t) = \frac{e^{-\delta(t)}}{\int_t^T e^{-\delta(s)} ds + K e^{-\delta(T)}} = \lambda_h^*(t).$$

Substituting these equalities into the wealth process equation and by Theorem 4.5, we can express the optimal wealth process of the informed agent in terms of the optimal wealth process of the uninformed agent :

$$X_i^{\left(c_{\lambda_i^*},\pi_i^*\right)}(t) = X_h^{\left(c_{\lambda_h^*},\pi_h^*\right)}(t)\exp\left\{\int_0^t\left(-\frac{1}{2}\zeta(s)^2+\zeta(s)\alpha(s)\right)ds\right.$$
$$+\int_0^t\int_{\mathbb{R}_0}\ln\left(1+\frac{z\zeta(s)\sigma(s)}{\sigma(s)^2+(\mu(s)-r(s)-\rho)z}\right)\nu_{\mathcal{G}}(ds,dz)+\int_0^t\zeta(s)d\hat{B}(s)$$
$$\left.+\int_0^t\int_{\mathbb{R}_0}\ln\left(1+\frac{z\zeta(s)\sigma(s)}{\sigma(s)^2+(\mu(s)-r(s)-\rho)z}\right)(N-\nu_{\mathcal{G}})(ds,dz)\right\}.$$

Hence,

$$J_i\left(c_{\lambda_i^*},\pi_i^*\right) = J_h\left(c_{\lambda_h^*},\pi_h^*\right)+\mathbb{E}\left[\int_0^T e^{-\delta(t)}\int_0^t\left(-\frac{1}{2}\zeta(s)^2+\zeta(s)\alpha(s)\right)ds\,dt\right]$$
$$+\mathbb{E}\left[\int_0^T e^{-\delta(t)}\int_0^t\int_{\mathbb{R}_0}\ln\left(1+\frac{z\zeta(s)\sigma(s)}{\sigma(s)^2+(\mu(s)-r(s)-\rho)z}\right)\nu_{\mathcal{G}}(ds,dz)\,dt\right]$$
$$+K\mathbb{E}\left[e^{-\delta(T)}\int_0^T\left(-\frac{1}{2}\zeta(s)^2+\zeta(s)\alpha(s)\right)ds\right]$$
$$+K\mathbb{E}\left[e^{-\delta(T)}\int_0^T\int_{\mathbb{R}_0}\ln\left(1+\frac{z\zeta(s)\sigma(s)}{\sigma(s)^2+(\mu(s)-r(s)-\rho)z}\right)\nu_{\mathcal{G}}(ds,dz)\right].$$

Acknowledgment. We are grateful to our supervisors B. Øksendal, N. Privault, and G. Di Nunno for the useful comments and suggestions. We also thank to M. Pontier and G. W. Weber for their fruitful remarks and suggestions. We wish to thank the *Advanced Mathematical Methods for Finance (AMaMeF)* programme of the *European Science Foundation (ESF)* for the financial supports.

References

1. Amendinger, J., Imkeller, P., and Schweizer, M.: Additional logarithmic utility of an insider, *Stochastic Processes and Their Applications*, **75** (1998) 263–286.
2. Biagini, F. and Øksendal, B.: A general stochastic calculus approach to insider trading, *Applied Mathematics and Optimization*, **52** (2005) 167–181.
3. Biagini, F. and Øksendal, B.: Minimal variance hedging for insider trading, *International Journal of Theoretical and Applied Finance*, **9** (2006) 1351–1375.
4. Cox, J. C. and Huang, C.: Optimal consumption and portfolio policies when asset prices follow a diffusion process, *Journal of Economic Theory*, **49** (1989), 33–83.
5. Cox, J. C. and Huang, C.: A variational problem arising in financial economics, *Journal of Mathematical Economics*, **20** (1991) 465–487.
6. Di Nunno, G., Meyer-Brandis, T., Øksendal, B., and Proske, F.: Malliavin Calculus for Lévy processes, *Infinite Dimensional Analysis, Quantum Probability and Related Fields*, **8** (2005) 235–258.
7. Di Nunno, G., Meyer-Brandis, T., Øksendal, B., and Proske, F.: Optimal portfolio for an insider in a market driven by Lévy processes, *Quantitative Finance*, **6(1)** (2006) 83–94.
8. Elliott, R. J., Geman, H., and Korkie, B. M.: Portfolio optimization and contingent claim pricing with differential information, *Stochastics and Stochastics Reports*, **60** (1997) 185–203.
9. Elliott, R. J. and Jeanblanc, M.: Incomplete markets with jumps and informed agents, *Math. Meth. Oper. Res*, **50** (1999) 475–492.

10. Grorud, A.: Asymmetric information in a financial market with jumps, *Inter. J. Theor. Appl. Fin.*, **3** (2000) 641–659.
11. Grorud, A. and Pontier, M.: Insider trading in a continuous time market model *International Journal of Theoretical and Applied Finance*, **1** (1998) 331–347.
12. He, H. and Pearson, N. D.: Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite-dimensional case, *Journal of Economic Theory*, **54** (1991) 259–304.
13. Imkeller, P.: Malliavin's calculus in insider models: additional utility and free lunches, *Mathematical Finance*, **13(1)** (2003) 153–169.
14. Jeulin, T.: *Semi-Martingales et Grossissement d'une Filtration*, Lecture Notes in Mathematics, vol. 833, Springer-Verlag, Berlin, Heidelberg, New York, 1980.
15. Karatzas, I., Lehoczky, J., and Shreve, S.: Optimal portfolio and consumption decisions for a 'small' investor on a finite horizon, *SIAM Journal of Control and Optimization*, **25** (1987) 1557–1586.
16. Karatzas, I., Lehoczky, J., Shreve, S., and Xu, G.: Martingale and duality methods for utility maximization in an incomplete market, *SIAM Journal of Control and Optimization*, **29(3)** (1991) 702–730.
17. Karatzas, I. and Pikovsky, I.: Anticipating portfolio optimization, *Advances in Applied Probability*, **28** (1996) 1095–1122.
18. Karatzas, I. and Zitkovic, G.: Optimal consumption from investment and random endowment in incomplete semimartingale markets, *The Annals of Probability*, **31(4)** (2003) 1821–1858.
19. Kohatsu-Higa, A. and Sulem, A.: Utility maximization in an insider influenced market, *Mathematical Finance*, **16(1)** (2006) 153–179.
20. Kramkov, D. and Schachermayer, W.: A condition on the asymptotic elasticity of utlity functions and optimal investment in incomplete markets, *The Annals of Applied Probability*, **9** (1999) 904–950.
21. León J. A., Navarro, R., and Nualart, D.: An anticipating calculus approach to the utility maximization of an insider, *Mathematical Finance*, **13(1)** (2003) 171–185.
22. Merton R. C.: Lifetime portfolio selection under uncertainity, *Review of Economics and Statistics*, **51** (1969) 247–257.
23. Merton R. C.: Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory*, **3** (1971) 373–413.
24. Nualart D.: Non causal stochastic integrals and calculus, *Stochastic Analysis and Related Topics, Proceedings Silivri*, eds. H. Körezlioğlu, S. Üstünel, **1316** (1986) 80–129.
25. Øksendal B.: A universal optimal consumption rate for an insider, *Mathematical Finance*, **16** (2006) 119–129.
26. Pliska S. R.: A stochastic calculus model of continuous trading: optimal portfolio, *Mathematics of Operations Research*, **11(2)** (1986) 371–382.
27. Russo F. and Vallois, P.: Forward, backward and symmetric stochastic integration, *Probability Theory Related Fields*, **9** (1993) 403–421.
28. Russo F. and Vallois, P.: Stochastic calculus with respect to continuous finite quadratic variation processes, *Stochastics and Stochastics Reports*, **70** (2000) 1–40.
29. Russo F. and Vallois, P.: *Elements of stochastic calculus via regularisation*, Séminaire de Probabilités XL, Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 2007.
30. Samuelson P. A.: Lifetime portfolio selection by dynamic stochastic programming, *The Review of Economics and Statistics*, **51** (1969) 239–246.

Delphine David: Laboratoire d'Analyse et Probabilités, Université d'Evry-Val-d'Essonne, France
E-mail address: delphine.david@univ-evry.fr
URL: http://perso.univ-lr.fr/ddavid/Home.html

Yeliz Yolcu Okur: Centre of Mathematics for Applications (CMA), University of Oslo, Norway
E-mail address: yelizy@math.uio.no